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AN APPROACH TO THE PROBLEM OF STABILIZING SYSTEMS WITH DELAY†

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The problem of stabilizing linear steady systems with delay when they are acted upon by various types of linear feedback is considered. Particular attention is given to the stabilization of second-order systems with feedback in the form of difference controllers. When the construction of a difference controller is complicated, a linear integral feedback is proposed for solving the problem. Unlike the well-known Krasovskii–Osipov method of constructing integral feedback by solving a linear-quadratic problem, the proposed method is based on the Wiener–Paley theorem for exponential-type integral functions. Copyright © 1996 Elsevier Science Ltd.

The problem of stabilizing dynamic systems with delay was considered for the first time by Krasovskii and Osipov [1, 2]. To solve it they introduced integral-type linear feedback, to construct which they used a method based on Lyapunov–Krasovskii functionals and a Shimanov bilinear form [3]. However, to use this method in practice it is necessary to find eigenvalues and eigenfunctions of the system, which is itself a special serious problem. It is therefore of interest to consider linear feedback in the form of difference controllers [4–6], which are much easier to realize in practice than integral-type controllers.

Below, using the example of two-dimensional systems, we present constructive algorithms for designing difference-type controllers, based on the sufficient conditions for stabilization, and which do not require a knowledge of the characteristic values. We consider a "scale" of feedback-type controllers and a new method (based on the Wiener–Paley theorem from the theory of integral functions of finite degree) for constructing an integral controller. It is assumed implicitly that the phase space of the systems considered with aftereffect is $C[-h, 0]$. From the point of view of the "scale" of the feedback controllers, the specific form of the phase space is not important (when considering the state space of a system with aftereffect, more accurately, the space of "minimum" states, this question is undoubtedly important). Hence, in this paper we did not put any particular emphasis on the specific form of the phase space.

We begin the investigations with a simpler controller, and, in the case when there is no stabilizing controller in this class or if its construction is complicated, we consider linear feedback that is next in complexity until we obtain an appropriate stabilizing controller (or we establish that the system cannot be stabilized). It is shown that in the well-known Krasovskii–Osipov example [1], in addition to the proposed integral-type controller, one can also construct a difference-type controller.

1. FORMS OF LINEAR FEEDBACK FOR SYSTEMS WITH AFTEREFFECT

Consider a system with a delaying argument

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + bu(t), \quad t > 0 \tag{1.1}$$
$$x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}, \quad t > 0; \quad b \in \mathbb{R}^n$$

where A and A_1 are matrices of appropriate dimensions and $h > 0$ is a constant delay.

In the problem of stabilizing such a system, integral feedback has been considered [1] in the form

$$u(t) = \int_{-h}^{0} [dQ(s)]x(t+s), \quad t > 0 \tag{1.2}$$

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where $Q(s)$ is a $1 \times n$ matrix function, the components of which are functions of limited variation in the range $[-h, 0]$.

To solve the modal-control problem the more general feedback

$$u(t) = \int_{-\theta}^{0} [dQ(s)]x(t+s), \quad t > 0 \tag{1.3}$$

was considered in [7], where $\theta \geqslant h$ is to be determined. When the Stieltjes measure in (1.3) is discrete and concentrated at the points $-jh$ ($j = 0, \ldots, N$), we obtain linear feedback in the form of a difference controller

$$u(t) = \sum_{j=0}^{N} q_j' x(t - jh), \quad q_j \in \mathbb{R}, \quad j = 0, 1, \ldots, N \tag{1.4}$$

(N is a natural number, and the prime indicates transposition). The controller (1.4) is more convenient to construct in practice. In this connection, the special case

$$u(t) = q_0' x(t) + q_1' x(t - h) \tag{1.5}$$

of controller (1.4) is of particular interest, since it does not take the closed system outside the limits of the class considered.

Controllers of the type (1.4) have been considered for systems with delay when investigating the problem of stabilization independently of the delay [6, 8]. The sufficient condition for such stabilization was effectively examined in [8], but it is somewhat restrictive. This condition was then refined in [6], but it was expressed in implicit form.

We note the controller [9]

$$\sum_{j=0}^{N} p_j u(t - jh) = \sum_{j=0}^{N} q_j' x(t - jh) \tag{1.6}$$

and its extension

$$-\int_{-\theta}^{0} dP(s)u(t+s) = -\int_{-\theta}^{0} dQ(s)x(t+s)$$

which may be useful for solving different problems of qualitative control theory in systems with aftereffect.

The approach to the stabilization problem proposed below can be extended to the system

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + D\dot{x}(t-h) + bu(t)$$

with a neutral type of delaying argument when acted upon by linear feedback of the form

$$u(t) = q_0' x(t) + q_1' x(t-h) + q_2' \dot{x}(t-h)$$

The use of different types of controllers may also be observed from [10–12].

2. FORMULATION OF THE PROBLEM AND FUNDAMENTAL RESULTS

System (1.1)–(1.3) is called a modally controlled system, if for any real numbers r_{ij} ($i = 1, \ldots, n; j = 0, \ldots, i$) there is a non-negative number θ and a matrix function $Q(\cdot)$ of limited variation in the interval $[-\theta, 0]$ such that the characteristic equation of system (1.1), closed by controller (1.3), has the form

$$\det\left[\lambda I - A - e^{-\lambda h} A_1 - b \int_{-\theta}^{0} e^{\lambda s} dQ(s)\right] = \lambda^n + \sum_{i=1}^{n} \sum_{j=0}^{i} r_{ij} \lambda^{n-i} e^{-\lambda jh} = 0, \quad \lambda \in \mathbb{C} \tag{2.1}$$

($\mathbb{C}$ is the field of complex numbers and I is the $n \times n$ identity matrix).

System (1.1), (1.3) is assumed to be stabilized if a controller of the form (1.3) exists for which the roots of the characteristic equation of the closed system have negative real parts.

We can similarly formulate the problem of modal control and stabilization for controllers (1.2) and (1.4)–(1.6).

The closed system (1.1), (1.6) is a neutral-type system.

We know [13] that system (1.1), (1.2) is stabilized if and only if the condition

$$\mathrm{rank} M(\lambda) = \mathrm{rank}[\lambda I - A - e^{-\lambda h}A_1, b\,] = n \tag{2.2}$$

is satisfied for any complex numbers λ, $\mathrm{Re}\,\lambda \geqslant 0$. The condition $\mathrm{rank}\, M(\lambda) = n$, $\forall\lambda \in \mathbb{C}$, is necessary and sufficient [7] for the modal controllability of system (1.1), (1.3).

Similarly [5], the condition

$$\det W(m) = \det[b, (A + mA_1)b, \ldots, (A + mA_1)^{n-1}] \equiv \mathrm{const} \neq 0, \quad m \in \mathbb{R} \tag{2.3}$$

is the criterion of modal controllability of system (1.1), (1.4). It is clear that condition (2.3) is sufficient to solve the problem of stabilizing system (1.1), (1.4). In more general form:† if the roots of the equation $\det W(m) = 0$ lie outside the circle $|\,m\,| \leqslant 1$, system (1.1), (1.6) can be stabilized. Condition (2.3) is necessary for the modal control of system (1.1) by controller (1.5), but, as the example shows

$$A = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad A_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}$$

($\lambda^2 - 1 = 0$ is the characteristic equation of the closed system), is not sufficient. The sufficient conditions are derived below for the problem of stabilizing second-order systems.

3. STABILIZATION OF LINEAR TWO-DIMENSIONAL SYSTEMS WITH DELAYING ARGUMENT

Consider system (1.1) with $n = 2$

$$A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}, \quad A_1 = \begin{Vmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{Vmatrix}, \quad b = \begin{Vmatrix} b_1 \\ b_2 \end{Vmatrix}, \quad b_1^2 + b_2^2 \neq 0 \tag{3.1}$$

and controller (1.5). We will write the feedback (1.5) in operator form

$$\begin{gathered} u(t) = [\beta_1(e^{-ph}), \quad \beta_2(e^{-ph})]x(t) \\ \beta_i(e^{-ph}) = \beta_{i0} + \beta_{i1}e^{-ph}, \quad i = 1,2 \\ \beta_{ij} \in \mathbb{R}, \quad i = 1,2, \quad j = 0,1; \quad [\beta_{10}, \beta_{20}] = q_0', \quad [\beta_{11}, \beta_{21}] = q_1' \end{gathered} \tag{3.2}$$

e^{-ph} is the delay operator and $e^{-ph}x(t) = x(t - h)$, $p = d/dt$.

Since condition (2.2) is necessary to stabilize system (1.1), (3.1), we will assume it is satisfied.

We put $\Delta = \det[b, Ab]$, $\Delta_1 = \det[b, A_1 b]$.

There are two possibilities: (1) $\Delta_1 = 0$, (2) $\Delta_1 \neq 0$.

We will consider case 1. The condition $\Delta_1 = 0$ means that b is an eigenvector of the matrix A_1, i.e. $A_1 b = \alpha_{22}^1 b$ ($\alpha_{22}^1 \in \mathbb{R}$). If b is also an eigenvector of the matrix A, i.e. $Ab = \alpha_{22} b$ ($\alpha_{22} \in \mathbb{R}$), the transformation $x = Ty$, $T = [d, b]$ with arbitrary vector d, such that $\det T \neq 0$, reduces system (1.1) to the form

†KIRILLOVA F. M. and MARCHENKO V. M., Functional transformations and certain canonical forms in linear systems with delaying arguments. Preprint No. 7(39), Inst. Mat. Akad. Nauk BSSR, Minsk, 1978.

$$\dot{y}(t) = \begin{Vmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} y(t) + \begin{Vmatrix} \alpha_{11}^1 & 0 \\ \alpha_{21}^1 & \alpha_{22}^1 \end{Vmatrix} y(t-h) + \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} u(t) \tag{3.3}$$

where $\alpha_{11}, \alpha_{21}, \alpha_{11}^1, \alpha_{22}^1$ are certain real numbers.

Requirement (2.2) then reduces to the condition

$$\lambda - \alpha_{11} - \alpha_{11}^1 e^{-\lambda h} \neq 0 \tag{3.4}$$

for all complex numbers λ, $\operatorname{Re} \lambda \geqslant 0$.

We will need for the following lemma below.

Lemma [14, pp. 146–147]. Suppose α and β are real numbers. Then the roots of the question

$$\lambda + \alpha + \beta e^{-\lambda h} = 0 \tag{3.5}$$

have only negative real parts if and only if the point (α, β) belongs to the stability domain Ω, the boundary of which is described by the lines (Fig. 1)

$$\beta = -\alpha, \quad \begin{cases} \alpha + \beta \cos hg = 0, \\ g - \beta \sin hg = 0, \end{cases} \quad 0 < g < \frac{\pi}{h} \ (L) \tag{3.6}$$

Moreover, if the point (α, β) lies in the domain Ω_1 (Fig. 2), the assertion of the lemma holds for all $h > 0$.

Assertion 1. Condition (3.4) is sufficient for system (1.1) to be made stable with controller (1.5).

Proof. Suppose $s_{10}, s_{20}, s_{11}, s_{21}$ are real numbers such that the equation

$$\lambda - s_{20} - s_{21} e^{-\lambda h} = 0 \tag{3.7}$$

has no roots with non-negative real parts, for example, $s_{20} < 0, s_{21} = 0$. Then the characteristic equation of system (1.1), closed by the controller

$$u(t) = [-\alpha_{21} + s_{10}, -\alpha_{22} + s_{20}] y(t) + [-\alpha_{21}^1 + s_{11}, -\alpha_{22}^1 + s_{21}] y(t-h) \tag{3.8}$$

has the form $(\lambda - \alpha_{11} - \alpha_{11}^1 e^{-\lambda h})(\lambda - s_{20} - s_{21} e^{-\lambda h}) = 0$ and, thus, has no roots in the left half-plane. Hence, system (3.3) and, consequently, (1.1) also, in this case can be stabilized by controller (1.5).

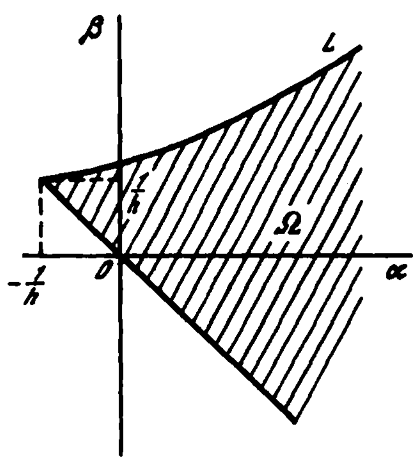

Fig. 1.

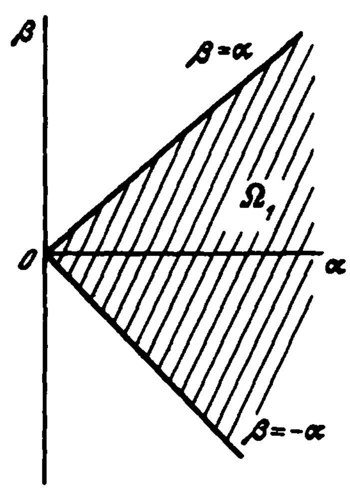

Fig. 2.

We therefore have the following theorem.

Theorem 1. When $\Delta = 0$, $\Delta_1 = 0$, system (1.1) can be stabilized by controller (1.5) when and only when the point $(-\alpha_{11}, -\alpha_{11}^1)$ from (3.3) lies in the domain Ω. System (1.1) can be stabilized for all delays $h > 0$ when and only when the point $(-\alpha_{11}, -\alpha_{11}^1)$ lies in the domain Ω_1.

Theorem 1 follows from Assertion 1, if we take into account the lemma and the necessary condition for stabilizability.

Note 1. As follows from the proof of Assertion 1, a stabilizing controller can be chosen in the form (3.8), where the numbers α_{11} and α_{11}^1 are the eigenvalues of the matrices A and A_1, respectively, which correspond to eigenvectors of these matrices different from b.

Suppose further that $\Delta_1 = 0$, $\Delta \neq 0$. Carrying out the transformation $x = Ty$, where $T = [Ab - (a_{11} + a_{22})b, b]$, we can reduce system (1.1) to the form

$$\dot{y}(t) = \begin{Vmatrix} 0 & 1 \\ -r_{20} & -r_{21} \end{Vmatrix} y(t) + \begin{Vmatrix} \alpha_{11}^1 & 0 \\ \alpha_{21}^1 & \alpha_{22}^1 \end{Vmatrix} y(t-h) + \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} u(t) \tag{3.9}$$

(r_{10}, r_{20}, α_{11}^1, α_{21}^1, α_{22}^1 are certain real numbers). Assuming

$$u(t) = [r_{20}, r_{10}]y(t) - [\alpha_{21}^1, \alpha_{22}^1]y(t-h) + v(t) \tag{3.10}$$

we obtain

$$\dot{y}(t) = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} y(t) + \begin{Vmatrix} \alpha_{11}^1 & 0 \\ 0 & 0 \end{Vmatrix} y(t-h) + \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} v(t) \tag{3.11}$$

Closing system (1.1) with the feedback

$$\begin{gathered} v(t) = [\eta_1(e^{-ph}),\ \eta_2(e^{-ph})]y(t), \quad p = \frac{d}{dt} \\ \eta_i(e^{-ph}) = \eta_{i0} + \eta_{i1}e^{-ph}, \quad \eta_{ij} \in \mathbf{R}, \quad i = 1,2, \quad j = 0,1 \end{gathered} \tag{3.12}$$

we arrive at the following characteristic quasi-polynomial

$$\det\begin{Vmatrix} \lambda - \alpha_{11}^1 e^{-\lambda h} & -1 \\ -\eta_1(e^{-\lambda h}) & \lambda - \eta_2(e^{-\lambda h}) \end{Vmatrix} =$$

$$= \lambda^2 - \lambda(\alpha_{11}^1 e^{-\lambda h} + \eta_2(e^{-\lambda h})) + \alpha_{11}^1 e^{-\lambda h}\eta_2(e^{-\lambda h}) - \eta_1(e^{-\lambda h}) \operatorname{def} \Theta$$

of the closed system.

Suppose λ_0, α, β are arbitrary real numbers such that $\lambda_0 > 0$, $(\alpha, \beta) \in \Omega$ (for example, $\alpha > |\alpha_{11}^1|$). Then the equation $(\lambda + \lambda_0)(\lambda + \alpha + \beta e^{-\lambda h}) = 0$ has no roots with negative real part. We will now require that

$$\Theta = (\lambda + \lambda_0)(\lambda + \alpha + \beta e^{-\lambda h}) = \lambda^2 + \lambda(\alpha + \lambda_0 + \beta e^{-\lambda h}) + \lambda_0(\alpha + \beta e^{-\lambda h})$$

For this it is sufficient to put

$$\begin{gathered} \alpha_{11}^1 e^{-\lambda h} + \eta_2(e^{-\lambda h}) = -\alpha - \lambda_0 - \beta e^{-\lambda h} \\ \alpha_{11}^1 e^{-\lambda h}\eta_2(e^{-\lambda h}) - \eta_1(e^{-\lambda h}) = \lambda_0(\alpha + \beta e^{-\lambda h}) \end{gathered}$$

whence it follows that

$$\eta_1(e^{-\lambda h}) = -\lambda_0(\alpha + \beta e^{-\lambda h}) - \alpha_{11}^1 e^{-\lambda h}(\alpha + \lambda_0 + \beta e^{-\lambda h} + \alpha_{11}^1 e^{-\lambda h})$$
$$\eta_2(e^{-\lambda h}) = -\alpha - \lambda_0 - \beta e^{-\lambda h} - \alpha_{11} e^{-\lambda h}$$

Suppose further that $\beta + \alpha_{11} = 0$. Then

$$\eta_1(e^{-\lambda h}) = -\alpha\lambda_0 - \alpha_{11}^1 \alpha e^{-\lambda h}, \quad \eta_2(e^{-\lambda h}) = -\alpha - \lambda_0 \tag{3.13}$$

Hence, by (3.10), (3.12) and (3.13) the required controller can be chosen in the form

$$u(t) = [r_{20} - \alpha\lambda_0, r_{10} - \alpha - \lambda_0]y(t) - [\alpha_{21}^1 + \alpha\alpha_{11}^1, \alpha_{22}^1]y(t-h) \tag{3.14}$$

Note that, by virtue of the lemma when $\alpha > |\alpha_{11}^1|$, controller (3.14) guarantees a stable closed system for all $h > 0$.

Theorem 2. If $\Delta \neq 0$, $\Delta_1 = 0$, system (1.1), (3.1) can be stabilized by feedback (1.5) for any delay h, $h > 0$. Then the stabilizing controller is defined by relation (3.14), taking the inverse transformation $y = T^{-1}x$ into account.

Consider case 2: $\Delta_1 \neq 0$. Using the transformation $x = Ty$, $T = [A_1 b + r_{11} b, b]$ we obtain

$$\dot{y}(t) = \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix} y(t) + \begin{Vmatrix} 0 & 1 \\ -r_{22} & -r_{11} \end{Vmatrix} y(t-h) + \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} u(t) \tag{3.15}$$

where r_{11} and r_{22} are defined in (2.1). Assuming that

$$u(t) = [-\alpha_{21}, -\alpha_{22}]y(t) + [r_{22}, r_{11}]y(t-h) + v(t) \tag{3.16}$$

and choosing $v(t)$ in the same way as (3.12), we arrive at the following characteristic quasi-polynomial

$$\det\begin{Vmatrix} \lambda - \alpha_{11} & -\alpha_{12} - e^{-\lambda h} \\ -\eta_1(e^{-\lambda h}) & \lambda - \eta_2(e^{-\lambda h}) \end{Vmatrix} =$$
$$= \lambda^2 + \lambda(-\alpha_{11} - \eta_2(e^{-\lambda h})) + \alpha_{11}\eta_2(e^{-\lambda h}) - \eta_1(e^{-\lambda h})(\alpha_{12} + e^{-\lambda h}) \underset{\text{def}}{=} \oplus$$

of the closed system. When $\alpha_{11} < 0$, the system can be stabilized by feedback (3.12), (3.16) when $\eta_1(e^{-ph}) = 0$ and by an appropriate choice of $\eta_2(e^{-ph})$. We require that

$$\oplus \equiv (\lambda + \alpha_1 + \beta_1 e^{-\lambda h})(\lambda + \alpha_2 + \beta e^{-\lambda h}) \equiv$$
$$\equiv \lambda^2 + \lambda(\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)e^{-\lambda h}) + (\alpha_1 + \beta_1 e^{-\lambda h})(\alpha_2 + \beta_2 e^{-\lambda h}), \quad \lambda \equiv \mathbb{C}$$

where $(\alpha_1, \beta_1) \in \Omega$, $(\alpha_2, \beta_2) \in \Omega$. Hence

$$\eta_2 = -\alpha_{11} - \alpha_1 - \alpha_2 - (\beta_1 + \beta_2)e^{-\lambda h} \tag{3.17}$$

$$\eta_1 = \frac{-\alpha_{11}(\alpha_{11} + \alpha_1 + \alpha_2 + (\beta_1 + \beta_2)e^{-\lambda h})}{\alpha_{12} + e^{-\lambda h}} + \frac{-(\alpha_1 + \beta e^{-\lambda h})(\alpha_2 + \beta_2 e^{-\lambda h})}{\alpha_{12} + e^{-\lambda h}}, \quad \lambda \in \mathbb{C}$$

Since the expression $\eta_1(e^{-\lambda h})$ must be quasi-polynomial, we require that

$$-\alpha_{11}(\alpha_{11} + \alpha_1 + \alpha_2 - (\beta_1 + \beta_2)\alpha_{12}) - (\alpha_1 - \alpha_{12}\beta_1)(\alpha_2 - \alpha_{12}\beta_2) = 0 \tag{3.18}$$

Bearing in mind the fact that $(\alpha_1, \beta_1) \in \Omega$, $(\alpha_2, \beta_2) \in \Omega$ we conclude that $\alpha_{11} < 1/h$ when $\alpha_{12} = 0$. In this case the stabilizing controller can be chosen in the form

$$u(t)=[-\alpha_{21}-\alpha_{11}\beta_1-\alpha_{11}\beta_2-\alpha_1\beta_2-\alpha_2\beta_1,\ -\alpha_{22}-\alpha_{11}-\alpha_1-\alpha_2]y(t)+ \qquad (3.19)$$
$$+[r_{22}-\beta_1\beta_2,\ r_{11}-\beta_1-\beta_2]y(t-h)$$

Consider the case when $\alpha_{12}\neq 0$. From (3.18) we obtain $\beta_1=(\alpha_{11}+\alpha_1)/\alpha_{12}$. Taking into account the fact that $(\alpha_1,\beta_1)\in\Omega$, $(\alpha_2,\beta_2)\in\Omega$, we have

$$\alpha_1=-\frac{g\cos hg}{\sin hg}$$
$$\frac{g\cos hg}{\sin hg}<\frac{\alpha_{11}+\alpha_1}{\alpha_{12}}<\frac{g}{\sin hg},\quad 0<g<\frac{\tilde{n}}{h},\quad (\alpha_2,\beta_2)\in\Omega \qquad (3.20)$$

Since $\sin hg>0$ when $0<g<\tilde{n}/h$, inequality (3.20) can be represented in the equivalent form

$$\begin{cases}\alpha_{12}>0\\ -\alpha_{12}g<g\cos hg-\\ -\alpha_{11}\sin hg<-\alpha_{12}g\cos hg\end{cases}\quad\text{or}\quad\begin{cases}\alpha_{12}<0\\ -\alpha_{12}g\cos hg<g\cos hg-\\ -\alpha_{11}\sin hg<-\alpha_{12}g\end{cases} \qquad (3.21)$$

We fix g, $0<g<\tilde{n}/(2h)$. We then obtain from (3.21)

$$\begin{cases}\alpha_{12}>\alpha_{11}\sin hg/g-\cos hg\\ \alpha_{12}<\alpha_{11}\,\mathrm{tg}\,hg/g-1\\ \alpha_{12}>0\end{cases}\quad\text{or}\quad\begin{cases}\alpha_{12}<\alpha_{11}\sin hg/g-\cos hg\\ \alpha_{12}>\alpha_{11}\,\mathrm{tg}\,hg/g-1\\ \alpha_{12}<0\end{cases}$$

(Fig. 3).

If $g\to\tilde{n}/(2h)$, it follows from (3.21) that $(\alpha_{11}\geqslant 0)$ $\alpha_{12}>2h\alpha_{11}/\tilde{n}$, $\alpha_{11}>0$ (Fig. 4).

We now fix g, $\tilde{n}/(2h)<g<\tilde{n}/h$. From (3.21) we have $(\alpha_{11}\geqslant 0)$

$$\begin{cases}\alpha_{12}>\alpha_{11}\sin hg/g-\cos hg\\ \alpha_{12}>\alpha_{11}\,\mathrm{tg}\,hg/g-1\\ \alpha_{12}>0\end{cases}\quad\text{or}\quad\begin{cases}\alpha_{12}<\alpha_{11}\sin hg/g-\cos hg\\ \alpha_{12}<\alpha_{11}\,\mathrm{tg}\,hg/g-1\\ \alpha_{12}<0\end{cases}$$

(Fig. 5).

For the point $(0,-1)$ criterion (2.2) is not satisfied, and hence when $\alpha_{11}=0$ and $\alpha_{12}=-1$ the system is not stabilizable.

Analysing the change in the parameter g from 0 to $\tilde{n}/h$ (Figs 3–5), we obtain the domain Ω_2 in which system (1.1), (3.15) can be stabilized by controller (1.5) (Fig. 6). This is an open domain, bounded by the straight lines $\alpha_{12}=1$, $\alpha_{12}=-1$, $\alpha_{12}=\alpha_{11}h-1$.

Hence we have the following theorem.

Theorem 3. Suppose $\Delta_1\neq 0$. Then system (1.1), (3.15) can be stabilized by controller (1.5) if the point $(\alpha_{11},\alpha_{12})$ lies in the domain Ω_2.

When $\alpha_{12}\neq 0$, $\alpha_{11}\geqslant 0$ a stabilizing controller can be chosen in a form similar to (3.19) (with r_{11} and r_{22} replaced by r_{11}^1, r_{22}^1).

When $(\alpha_{11},\alpha_{12})\notin\Omega_2$ the question of whether system (1.1), (3.15), (1.5) can be stabilized remains open. But if $e^{-\alpha_{11}h}+\alpha_{12}\neq 0$ for $\alpha_{11}\geqslant 0$, an integral controller exists (see (2.2)) which solves the stabilization problem. Below we propose a simple way of constructing such a controller which does not require the eigenvectors of the system to be determined (unlike the existing Krasovskii–Osipov method [1, 2]).

Thus, we will consider system (3.15), (3.16)

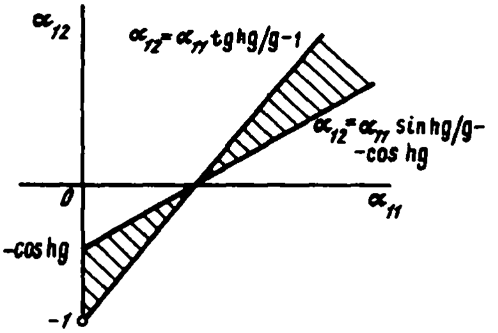

Fig. 3.

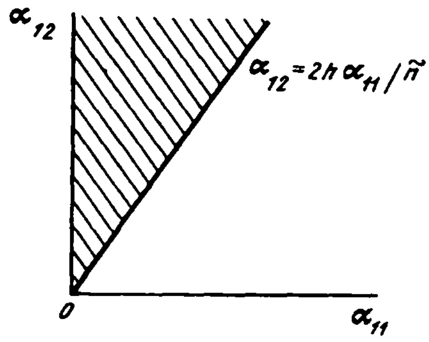

Fig. 4.

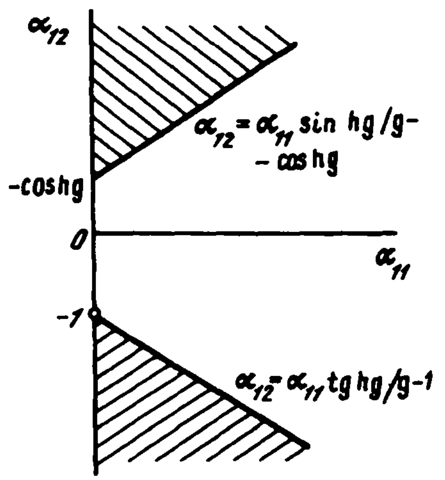

Fig. 5.

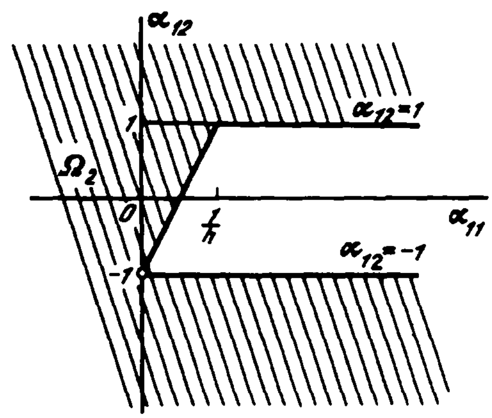

Fig. 6.

$$\dot{y}(t)=\begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ 0 & 0 \end{Vmatrix} y(t)+\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} y(t-h)+\begin{Vmatrix} 0 \\ 1 \end{Vmatrix} v(t) \tag{3.22}$$

and the linear integral feedback

$$v(t)=\int_{-h}^{0} dQ(s)y(t+s), \quad t>0 \tag{3.23}$$

or, in operator form

$$v(t)=\int_{-h}^{0} e^{ps}dQ(s)y(t) \underset{\text{def}}{=} [\eta_1(e^{-p}),\eta_2(e^{-p})]y(t), \quad t>0 \tag{3.24}$$

By virtue of the Wiener–Paley theorem, taking the form of the controller (1.2) into account, it is sufficient to seek the functions η_1 and η_2 in the class of linear combinations of polynomials of the first degree in e^{-ph} and integral functions which are quadratically integrable along the imaginary axis. Then, reverting to the originals, we obtain a controller of the form (3.23) [15].

We shall require that, for the characteristic quasi-polynomial of system (3.22), closed by the controller (3.24), the following relation holds

$$\det\begin{Vmatrix} p-\alpha_{11} & -\alpha_{12}-e^{-ph} \\ -\eta_1(e^{-p}) & p-\eta_2\ (e^{-p}) \end{Vmatrix} \equiv p^2+r_1p+r_2, \quad p\in\mathbb{C}$$

where $p^2 + r_1p + r_2$ is an arbitrary stable polynomial ($r_1 \in \mathbf{R}, r_2 \in \mathbf{R}$). As a result we have

$$\eta_2(e^{-p}) = -(r_1+\alpha_{11}) + \frac{\alpha_{11}(r_1+\alpha_{11})+r_2+\eta_1(e^{-p})(\alpha_{12}+e^{-ph})}{\alpha_{11}-p}$$

We will now choose $\eta_1(e^{-p})$ so that the following equation is satisfied

$$\alpha_{11}(r_1+\alpha_{11})+r_2+\eta_1(e^{-\alpha_{11}})(\alpha_{12}+e^{-\alpha_{11}h})=0$$

whence ($\alpha_{12} + e^{-\alpha_{11}h} \neq 0$ by virtue of (2.2))

$$\eta_1(e^{-\alpha_{11}}) = \frac{-\alpha_{11}r_1-\alpha_{11}-r_2}{\alpha_{12}+e^{-\alpha_{11}h}} = \eta_1^*$$

Assuming $\eta_1(e^{-p}) \equiv \eta_1^*$ we obtain

$$\eta_2(e^{-p}) = -(r_1+\alpha_{11}) + \frac{\alpha_{11}r_1+\alpha_{11}^2+r_2+\eta_1^*(\alpha_{12}+e^{-ph})}{\alpha_{11}-p} = -(r_1+\alpha_{11}) + \frac{\eta_2^*+\eta_1^*e^{-ph}}{\alpha_{11}-p},$$

$$\eta_2^* = \frac{e^{-\alpha_{11}h}(\alpha_{11}r_1+\alpha_{11}^2+r_2)}{\alpha_{12}+e^{-\alpha_{11}h}}$$

Reverting to the originals, we have

$$\frac{\eta_2^*+\eta_1^*e^{-ph}}{\alpha_{11}-p} \risingdotseq \begin{cases} -\eta_2^*e^{\alpha_{11}t}, & t\in[0,h] \underset{\text{def}}{=} q_2(t) \\ 0, & t>h \end{cases}$$

$$\frac{\eta_2^*+\eta_1^*e^{-ph}}{\alpha_{11}-p} y_2(p) \risingdotseq \int_0^t q_2(\tau)y_2(t-\tau)d\tau =$$

$$= \int_0^h \hat{q}_2(\tau)y_2(t-\tau)d\tau = \int_{-h}^0 q_2(-\mu)y_2(t+\mu)d\mu,$$

$$\hat{q}_2(\tau) = \begin{cases} q_2(\tau), & \tau\leqslant \max\{t,h\} \\ 0, & \tau>\max\{t,h\} \end{cases}$$

As a result we obtain the stabilizing controller in the form

$$v(t) = \frac{-\alpha_{11}r_1-\alpha_{11}^2-r_2}{\alpha_2+e^{-\alpha_{11}h}} y_1(t) - (r_1+\alpha_{11})y_2(t) + \int_{-h}^0 \hat{q}_2(-\mu)y_2(t+\mu)d\mu$$

4. EXAMPLE

Consider the following system [1]

$$\dot{x}(t) = \begin{Vmatrix} 0 & 0 \\ a & 0 \end{Vmatrix} x(t) + \begin{Vmatrix} -\frac{\tilde{n}}{2} & 0 \\ d & 0 \end{Vmatrix} x(t-1) + \begin{Vmatrix} b_1 \\ b_2 \end{Vmatrix} \xi$$

where a, d, b_1 and b_2 are constant parameters and ξ is the control. We have

$$\Delta = ab_1^2, \quad \Delta_1 = b_1b_3; \quad b_3 = db_1 + \tilde{n}b_2/2$$

Two situations are possible: (1) $b_1 = 0$, (2) $b_1 \neq 0$.

Suppose $b_1 = 0$. Then $\Delta = 0$, $\Delta_1 = 0$. Using the transformation

$$x = Ty, \quad T = \begin{Vmatrix} 1 & 0 \\ 0 & b_2 \end{Vmatrix}, \quad b_2 \neq 0$$

and assuming that

$$u(t) = [-a/b_2, 0]y(t) + [-d/b_2, 0]y(t-h) + v(t)$$

we have

$$\dot{y}(t) = \begin{Vmatrix} -\tilde{n}/2 & 0 \\ 0 & 0 \end{Vmatrix} y(t-h) + \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} v(t)$$

whence $\alpha_{11} = 0$, $\alpha_{11}^1 - \tilde{n}/2$. The point $(0, \tilde{n}/2) \notin \Omega$. Hence (by Theorem 1) the system in this case cannot be stabilized.

Suppose now that $b_3 = 0$, $b_1 \neq 0$, $a \neq 0$. Then (by Theorem 2) the system can be stabilized by the controller (1.5). The controller can be chosen so that

$$u(t) = [-\alpha\lambda_0, -\alpha - \lambda_0]y(t) + [0, \tilde{n}/2 + \tilde{n}\alpha/2]y(t-h)$$

where α and λ_0 are arbitrary positive real numbers.

When $b_1 \neq 0$, $b_3 \neq 0$, we use the transformation

$$x = Ty, \quad T = \begin{Vmatrix} 0 & b_1 \\ b_3 & b_2 \end{Vmatrix}$$

Assuming

$$u(t) = [0, \tilde{n}/2]y(t-h) + v(t)$$

we have

$$\dot{y}(t) = \begin{Vmatrix} 0 & b_4 \\ 0 & 0 \end{Vmatrix} y(t) + \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} y(t-h) + \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} v(t), \quad b_4 = b_1 a / b_3$$

whence $\alpha_{11} = 0$, $\alpha_{12} = b_4$.

The condition for stabilizability is $ab_1 + b_3 \neq 0$. If this is satisfied, the stabilizing controller (1.5) can be constructed as follows. If $a = 0$, we have

$$u(t) = [-\alpha_1\beta_2 - \alpha_2\beta_1, -\alpha_1 - \alpha_2]y(t) + [-\beta_1\beta_2, -\beta_1 - \beta_2 + \tilde{n}/2]y(t-h)$$
$$(\alpha_i, \beta_i) \in \Omega, \quad i = 1, 2$$

When $a \neq 0$ we choose the parameter g so that

(a) $g \in \left(\frac{\tilde{n}}{2h}, \frac{\tilde{n}}{h}\right)$, $\cos hg < -\alpha_{12}$ when $b_4 < -1$

(b) $g \in \left(0, \frac{\tilde{n}}{2h}\right)$, $\cos hg < -\alpha_{12}$ when $-1 < b_4 < 0$

(c) $g \in \left(\frac{\tilde{n}}{2h}, \frac{\tilde{n}}{h}\right)$, $\cos hg > -\alpha_{12}$ when $b_4 > 0$

Then the required controller is sought in the form (3.19) (with r_{11} and r_{22} replaced by r_{11}^1, r_{22}^1), where

$$\alpha_1 = -\frac{g\cos hg}{\sin hg}, \quad \beta_1 = \frac{\alpha_{11} + \alpha_1}{\alpha_{12}} = \frac{\alpha_1}{b_4}$$

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REFERENCES

1. KRASOVSKII N. N. and OSIPOV Yu. S., Stabilization of the motions of a controlled system with delay in a regulation system. *Izv. Akad. Nauk SSSR. Tekh. Kibernetika* 6, 3–15, 1963.
2. OSIPOV Yu. S., Stabilization of controlled systems with delay. *Diff. Uravneniya* **1**, 5, 605–618, 1965.
3. SHIMANOV S. N., The theory of linear differential systems with aftereffect. *Diff. Uravneniya* **1**, 1, 102–116, 1965.
4. MORSE A. S., Ring models for delay-differential systems. *Automatica* **12**, 5, 529–531, 1976.
5. ASMYKOVICH I. K. and MARCHENKO V. M., Control of the spectrum of systems with delay. *Avtomatika i Telemekhanika* 7, 5–14, 1976.
6. DATKO R., Remarks concerning the asymptotic stability and stabilization of linear delay differential equations. *Math. Anal. Appl.* **111**, 2, 571–584, 1985.
7. MARCHENKO V. M., The problem of modal control in linear systems with delay. *Dokl. Akad. Nauk BSSR* **22**, 5, 401–404, 1978.
8. KAMEN E. W., Linear systems with commensurate time delays: stability and stabilization independent of delay. *IEEE Trans. Aut. Control* **AC-27**, 2, 367–375, 1982.
9. MARCHENKO V. M., Modal control in systems with aftereffect. *Avtomatika i Telemekhanika* 11, 73–83, 1988.
10. PANDOLFI L., Stabilization of neutral functional differential equations. *J. Optimiz. Theory Appl.* **20**, 2, 191–204, 1976.
11. OLBROT A. W., Stabilizability, detectability and spectrum assignment for linear autonomous systems with general time delays. *IEEE Trans. Aut. Control* **AC-23**, 5, 887–890, 1978.
12. WATANABE F., Finite spectrum assignment of linear systems with a class of noncommensurate delays. *Int. J. Control.* **47**, 5, 1277–1289, 1988.
13. GABELAYA A. G., IVANENKO V. I. and ODARICH O. N., The stabilizability of linear autonomous systems with delay. *Avtomatika i Telemekhanika*, 8, 12–16, 1976.
14. EL'SGOL'TS L. E. and NORKIN S. B., *Introduction to the Theory of Differential Equations with Divergent Argument*. Nauka, Moscow, 1971.
15. SHILOV G. Ye., *Mathematical Analysis. A Special Course*. Fizmatgiz, Moscow, 1960.

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